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Killing forms on G_2- and Spin_7-manifolds

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Abstract

Killing forms on Riemannian manifolds are differential forms whose covariant derivative is totally skew-symmetric. We prove that on a compact manifold with holonomy G_2 or Spin_7 any Killing form has to be parallel. The main tool is a universal Weitzenböck formula. We show, how such a formula can be obtained for any given holonomy group and any representation defining a vector bundle.

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1. Introduction

Killing forms are a natural generalization of Killing vector fields. They are defined as differential forms u, such that ∇u is totally skew-symmetric. More generally one considers twistor forms, as forms in the kernel of an elliptic differential operator, defined similarly to the twistor operator in spin geometry. Twistor 1-forms are dual to conformal vector fields. Killing forms are coclosed twistor forms.

The notion of Killing forms was introduced by Yano in [17], where he already noted that a p-form u is a Killing form if and only if for any geodesic γ the $(p-1)$-form $\dot{\gamma} \lrcorner u$ is parallel along γ. In particular, Killing forms define quadratic first integrals of the geodesic equation, i.e. functions, which are constant along geodesics. This motivated an intense study of Killing forms in the physics literature, e.g. in the article of Penrose and Walker [12]. More recently, Killing and twistor forms have been successfully applied to define symmetries of field equations (cf. [4,5]).

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On the standard sphere, the space of twistor forms coincides with the eigenspace of the Laplace operator for the minimal eigenvalue and Killing forms are the coclosed minimal eigenforms. The sphere also realizes, the maximal possible number of twistor or Killing forms. So far, there are only very few further examples of compact manifolds admitting Killing p-forms with $p \geq 2$. These are Sasakian, nearly Kähler and weak-G_2 manifolds, and products of them. The Killing p-forms of these examples satisfy an additional equation and it turns out that they are in 1-1 correspondence to parallel $(p+1)$-forms on the metric cone. In particular, they only exist on manifolds with Killing spinors (cf. [2,13]).

The present article is the last step in the study of Killing forms on manifolds with restricted holonomy. It was already known that on compact Kähler manifolds Killing p-forms with $p \geq 2$ are parallel [16]. Moreover, we showed in Ref. [11,3] that the same is true on compact quaternion-Kähler manifolds and compact symmetric spaces. Here, we will prove the corresponding statement for the remaining holonomies G_2 and Spin_7.

The Hodge $*$-operator preserves the space of twistor forms. In particular, it maps Killing forms to closed twistor forms, which we will call $*$-Killing forms.

Theorem 1.1. *Let (M^7, g) be a compact manifold with holonomy G_2. Then, any Killing form and any $*$-Killing form is parallel. Moreover, any twistor p-form, with $p \neq 3, 4$, is parallel.*

Theorem 1.2. *Let (M^8, g) be a compact manifold with holonomy* Spin_7. *Then, any Killing form and any $*$-Killing form is parallel. Moreover, any twistor p-form, with $p \neq 3$–5, is parallel.*

The main tool for proving the two theorems are suitable Weitzenböck formulas for the irreducible components of the form bundle. More generally, we prove a universal Weitzenböck formula, i.e. we show, how to obtain for any fixed holonomy group G and any irreducible G-representation π, a Weitzenböck formula for certain first order differential operators acting on sections of the vector bundle defined by π. Our formula is already known in the case of Riemannian holonomy SO_n (cf. [8]). However, it seems to be new and so far unused in the case of the exceptional holonomies G_2 and Spin_7. We describe, here, an approach to Weitzenböck formulas which, is further developed and completed in Ref. [15].

2. Twistor forms on Riemannian manifolds

In this section, we recall the definition and basic facts on twistor and Killing forms. More details and further references can be found in Ref. [13]. Most important for the later application will be the integrability condition given in Proposition 2.2.

Consider a n-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$. Then, the tensor product $V^* \otimes \Lambda^p V^*$ has the following $O(n)$-invariant decomposition:

$$V^* \otimes \Lambda^p V^* \cong \Lambda^{p-1} V^* \oplus \Lambda^{p+1} V^* \oplus \Lambda^{p,1} V^*,$$

where $\Lambda^{p,1} V^*$ is the intersection of the kernels of wedge and inner product. This decomposition immediately translates to Riemannian manifolds (M^n, g), where we have:

$$T^*M \otimes \Lambda^p T^*M \cong \Lambda^{p-1} T^*M \oplus \Lambda^{p+1} T^*M \oplus \Lambda^{p,1} T^*M, \tag{1}$$

with $\Lambda^{p,1} T^*M$ denoting the vector bundle corresponding to the representation $\Lambda^{p,1}$. The covariant derivative $\nabla\psi$ of a p-form ψ is a section of $T^*M \otimes \Lambda^p T^*M$. Its projections onto the summands $\Lambda^{p+1} T^*M$ and $\Lambda^{p-1} T^*M$ are just the differential $\mathrm{d}\psi$ and the codifferential $\mathrm{d}^*\psi$. Its projection onto the third summand $\Lambda^{p,1} T^*M$ defines a natural first order differential operator T, called the *twistor operator*. The twistor operator $T : \Gamma(\Lambda^p T^*M) \to \Gamma(\Lambda^{p,1} T^*M) \subset \Gamma(T^*M \otimes \Lambda^p T^*M)$ is given for any vector field X by the following formula:

$$[T\psi](X) := [\mathrm{pr}_{\Lambda^{p,1}}(\nabla\psi)](X) = \nabla_X\psi - \frac{1}{p+1} X \lrcorner \, \mathrm{d}\psi + \frac{1}{n-p+1} X^* \wedge \mathrm{d}^*\psi. \qquad (2)$$

From now on, we will identify TM with T^*M using the metric.

Definition 2.1. A p-form ψ is called a twistor p-form if and only if ψ is in the kernel of T, i.e. if and only if ψ satisfies:

$$\nabla_X\psi = \frac{1}{p+1} X \lrcorner \, \mathrm{d}\psi - \frac{1}{n-p+1} X \wedge \mathrm{d}^*\psi, \qquad (3)$$

for all vector fields X. If the p-form ψ is in addition coclosed, it is called a Killing p-form. A closed twistor form is called $*$-Killing form.

Twistor forms are also known as *conformal Killing forms* or skew-symmetric *Killing-Yano tensors*. Twistor 1-forms are dual to conformal vector fields and Killing 1-forms are dual to Killing vector fields. Note that the Hodge star-operator $*$ maps twistor p-forms into twistor $(n-p)$-forms. In particular, it interchanges Killing and $*$-Killing forms.

Twistor forms are well understood on compact Kähler manifolds (cf. [10]). Here, they are closely related to Hamiltonian 2-forms recently studied in Ref. [1]. In particular, one has examples on the complex projective space in any even degree.

Differentiating Eq. (2), one obtains the two equations:

$$\nabla^*\nabla\psi = \frac{1}{p+1} \mathrm{d}^*\,\mathrm{d}\psi + \frac{1}{n-p+1} \mathrm{d}\,\mathrm{d}^*\psi + T^*T\psi, \qquad (4)$$

$$q(R)\psi = \frac{p}{p+1} \mathrm{d}^*\,\mathrm{d}\psi + \frac{n-p}{n-p+1} \mathrm{d}\,\mathrm{d}^*\psi - T^*T\psi, \qquad (5)$$

where $q(R)$ is the curvature term appearing in the classical Weitzenböck formula for the Laplacian on p-forms: $\Delta = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^* = \nabla^*\nabla + q(R)$. It is the symmetric endomorphism of the bundle of differential forms defined by:

$$q(R) = \sum e_j \wedge e_i \lrcorner \, R_{e_i,e_j}, \qquad (6)$$

where $\{e_i\}$ is any local orthonormal frame and R_{e_i,e_j} denotes the curvature of the form bundle. On 1-forms the endomorphism $q(R)$ is just the Ricci curvature. It is important to note that one may define $q(R)$ also in a more general context. For this, we first, rewrite Eq. (6) as:

$$q(R) = \sum_{i<j} (e_j \wedge e_i \lrcorner - e_i \wedge e_j \lrcorner) R_{e_i,e_j} = \sum_{i<j} (e_i \wedge e_j) R(e_i \wedge e_j),$$

where the Riemannian curvature R is considered as element of $\mathrm{Sym}^2(\Lambda^2 TM)$ and 2-forms act via the standard representation of the Lie algebra $\mathfrak{so}(T_mM) \cong \Lambda^2 T_mM$ on the space of p-forms. Note that we can replace $\{e_i \wedge e_j\}$ by any basis of $\mathfrak{so}(T_mM)$ orthonormal with respect to the scalar product induced by g on $\mathfrak{so}(T_mM) \cong \Lambda^2 T_mM$.

Let (M, g) be a Riemannian manifold with holonomy group $G = \mathrm{Hol}$. Then, the curvature tensor takes values in the Lie algebra $\mathfrak{g}$ of the holonomy group and we can write $q(R)$ as:

$$q(R) = \sum X_i R(X_i),$$

where $\{X_i\}$ is any orthonormal basis of $\mathfrak{g}$ acting via the form representation restricted to the holonomy group. It is clear that in this way $q(R)$ gives rise to a symmetric endomorphism on any associated vector bundle defined via a representation of the holonomy group. Moreover, this bundle endomorphism preserves any parallel sub-bundle and its action only depends on the representation defining the sub-bundle and not on the particular realization (cf. [13,14]).

Integrating Eq. (5), yields a characterization of twistor forms on compact manifolds. This, generalizes the characterization of Killing vector fields on compact manifolds, as divergence free vector fields in the kernel of $\Delta - 2\mathrm{Ric}$.

Proposition 2.2. *Let (M^n, g) a compact Riemannian manifold. Then, a p-form ψ is a twistor p-form, if and only if $q(R)\psi = \frac{p}{p+1}\,\mathrm{d}^*\,\mathrm{d}\psi + \frac{n-p}{n-p+1}\,\mathrm{d}\,\mathrm{d}^*\psi$. A coclosed p-form ψ is a Killing form if and only if $\nabla^*\nabla\psi = \frac{1}{p}q(R)\psi$. A closed p-form ψ is a $*$-Killing form if and only if $\nabla^*\nabla\psi = \frac{1}{n-p}q(R)\psi$. If $n = 2m$, then, a m-form ψ is a twistor form if and only if $\nabla^*\nabla\psi = \frac{1}{m}q(R)\psi$.*

For the later application, in the case of compact Ricci-flat manifolds, we still mention an immediate consequence of Eq. (5).

Corollary 2.3. *Let M be a compact manifold and let $EM \subset \Lambda^p T^*M$ be a parallel sub-bundle such that $q(R)$ acts trivially on EM. Then, any twistor and any harmonic form in EM has to be parallel.*

Manifolds with holonomy G_2 or Spin_7 are Ricci-flat. Hence, $q(R)$ vanishes on all bundles where it reduces to Ricci or scalar curvature.

Proposition 2.4. *Let (M^7, g) be a manifold with holonomy G_2. Then, $q(R)$ acts trivially on any bundle of rank less or equal to 7. Let (M^8, g) be a manifold with holonomy* Spin_7. *Then, $q(R)$ acts trivally on any bundle of rank less or equal to* 8.

Proof. Of course, $q(R)$ acts trivially on any bundle defined by a trivial representation. For the holonomy group G_2, the seven-dimensional holonomy representation is the smallest possible non-trivial representation. On the corresponding bundle $q(R)$ acts as Ricci curvature and, thus, vanishes.

For the holonomy group Spin_7, we have the eight-dimensional holonomy representation and the seven-dimensional standard representation as smallest possible non-trivial representations. In both cases, $q(R)$ acts trivially: the spinor bundle of a manifold with Spin_7-holonomy splits into the sum of a trivial line bundle, corresponding to the parallel spinor, and the sum of a bundle of rank 7 and a bundle of rank 8. These two bundles are induced by the eight-dimensional holonomy representation and by the seven-dimensional standard representation. It is well known that $q(R)$ acts as $\frac{s}{16}\mathrm{id}$ on the summands of the spinor bundle (cf. [14]). But, for Spin_7-manifolds the scalar curvature s is zero and we conclude that $q(R)$ acts trivially on the bundles in question. □

3. A universal Weitzenböck formula

In this section, we derive for any manifold with a fixed holonomy one basic Weitzenböck formula. The coefficients of this formula will depend on the holonomy group and the defining representation. This is only the first step of a more general method of producing all possible Weitzenböck formulas on such manifolds (cf. [15]).

We consider the following situation: let (M^n, g) be an oriented Riemannian manifold with holonomy group $G := \mathrm{Hol}(M, g) \subset \mathrm{SO}(n)$. Then, the $\mathrm{SO}(n)$-frame bundle reduces to a G-

principal bundle $P_G \to M$ and all natural bundles over M are associated to P_G via representations of G.

If $\pi : G \to \mathrm{Aut}(E)$ is a representation of G, we denote with EM the corresponding associated bundle over M. In particular, we will denote the holonomy representation of G, given by the inclusion $G \subset \mathrm{SO}(n)$, with T. The associated vector bundle is then, of course, the tangent bundle TM. For simplifying the notation, we will also write T for the complexified holonomy representation, which then defines the complexified tangent bundle TM.

The Levi-Civita connection of (M, g) induces a connection ∇ on any bundle EM and the covariant derivative of a section of EM is a section of $TM \otimes EM$. Hence, we may define natural first order differential operators by composing the covariant derivative with projections onto the components of $TM \otimes EM$. These operators are also known as *Stein-Weiss operators*.

Let $T \otimes E = \sum E_i$ be the decomposition of $T \otimes E$ into irreducible G-representations, where we consider E_i as a subspace of $T \otimes E$. This induces a corresponding decomposition of the tensor product $TM \otimes EM$. For any component E_iM, we define an operator T_i by:

$$T_i : \Gamma(EM) \to \Gamma(E_iM), \qquad T_i(\psi) := \mathrm{pr}_i(\nabla\psi),$$

where pr_i denotes the projection $T \otimes E \to E_i \subset T \otimes E$ and the corresponding bundle map. In the following, we will make no difference between representations resp. equivariant maps and the corresponding vector bundles resp. bundle maps.

Since we are on a Riemannian manifold, we have for any T_i its formal adjoint $T_i^* : \Gamma(E_iM) \to \Gamma(EM)$. The aim of this section is to derive a Weitzenböck formula for the second order operators $T_i^* \circ T_i$, i.e. a linear combination $\sum_i c_i T_i^* \circ T_i$ with real numbers c_i, which is of zero order, i.e. a curvature term. The coefficients c_i will depend on the holonomy group G and the representation E.

Our approach to Weitzenböck formulas, further developed in Ref. [15], is motivated by the following remarks. Let ψ be any section of EM, then, $\nabla^2\psi$ is a section of the bundle $TM \otimes TM \otimes EM$. Any G-equivariant homomorphism $F \in \mathrm{Hom}_G(T \otimes T \otimes E, E)$ defines by $\psi \mapsto F(\nabla^2\psi)$ a second order differential operator acting on sections of EM. For describing these homomorphisms, it is rather helpful to use the natural identifications:

$$\mathrm{Hom}_G(T \otimes T \otimes E, E) \cong \mathrm{End}_G(T \otimes E) \cong \mathrm{Hom}_G(T \otimes T, \mathrm{End}\, E).$$

A homomorphisms $F : T \otimes T \to \mathrm{End}\, E$ is mapped onto the endomorphism F of $T \otimes E$ defined by $F(a \otimes s) = \sum e_i \otimes F_{e_i \otimes a}(s)$, for any orthonormal basis $\{e_i\}$ of T and any $a \in T, s \in E$. Conversely, an endomorphism F is mapped to the homomorphism F with $F_{a\otimes b}(s) = a \lrcorner F(b \otimes s)$. In particular, the identity of $T \otimes E$ is mapped onto $\mathrm{id}_{a\otimes b} = g(a, b)\mathrm{id}_E$. Finally, $F \in \mathrm{Hom}(T \otimes T \otimes E, E)$ is defined as $F(a \otimes b \otimes s) = F_{a\otimes b}s$.

Beside the identity $\mathrm{id}_{T\otimes E}$, we have the projections $\mathrm{pr}_i : T \otimes E \to E_i \subset T \otimes E$ as important examples of invariant endomorphisms. The following proposition describes the corresponding second order differential operators.

Proposition 3.1. *Let $T \otimes E = \sum E_i$ the decomposition into irreducible summands, with corresponding operators T_i. Then, any section ψ of EM satisfies*:

$$\mathrm{id}(\nabla^2\psi) = -\nabla^*\nabla\psi, \tag{1'}$$

$$\mathrm{pr}_i(\nabla^2\psi) = -T_i^* T_i\,(\psi). \tag{2'}$$

Proof. Let $\{e_i\}$ be a parallel local ortho-normal frame of TM, then, $\nabla^2 = \sum e_i \otimes e_j \otimes \nabla_{e_i}\nabla_{e_j}$ and $\mathrm{id}(\nabla^2\psi) = \sum g(e_i, e_j)\nabla_{e_i}\nabla_{e_j}\psi = -\nabla^*\nabla\psi$, which proves Eq. (1′).

Eq. (2′)is a direct consequence of the following more general statement. □

Lemma 3.2. *Let (M, g) be a Riemannian manifold and let E, F be hermitian vector bundles over M, equipped with metric connections ∇. If $D : \Gamma(E) \to \Gamma(F)$ is a differential operator defined as $D = p \circ \nabla$, where $p : TM \otimes E \to F$ is some parallel linear map. Then, the adjoint operator for D is $D^* = \nabla^* \circ p^*$ and $D^*D = -\mathrm{tr} \circ (\mathrm{id} \otimes p^*p)\nabla^2$.*

The second equation follows with $F = TM \otimes EM$ and an orthogonal projection $p = \mathrm{pr}_i$ onto a sub-bundle $EM_i \subset TM \otimes EM$. Indeed, in this case we have $p^*p = p^2 = p$ and it follows $(T_i)T_i\psi = -\mathrm{tr} \circ (\mathrm{id} \otimes \mathrm{pr}_i)\nabla^2\psi = -\mathrm{pr}_i(\nabla^2\psi)$.

We will now prove the lemma. Note, that the formal adjoint ∇^* of the covariant derivative $\nabla : \Gamma(E) \to \Gamma(TM \otimes E)$ is given as the composition of the following differential operators: $\Gamma(TM \otimes E) \xrightarrow{\nabla} \Gamma(TM \otimes TM \otimes E) \xrightarrow{-\mathrm{tr}} \Gamma(E)$, where ∇ also denotes the tensor product connection, i.e. $\nabla := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$. Since D is defined as $D = p \circ \nabla$ we have $D^* = (p \circ \nabla)^* = \nabla^* \circ p^*$, thus, $D^*D = (p \circ \nabla)^*(p \circ \nabla) = \nabla^* \circ p^*p \circ \nabla$. Since p is a parallel map it commutes with ∇^*. Hence, we can substitute the formula for ∇^* to obtain $D^*D = \nabla^* \circ p^*p \circ \nabla = -\mathrm{tr} \circ \nabla \circ p^*p \circ \nabla = -\mathrm{tr} \circ (\mathrm{id} \otimes p^*p) \circ \nabla^2$.

Since we obviously have $\mathrm{id} = \sum \mathrm{pr}_i$, the proposition above immediately implies a rather useful formula for the operator $\nabla^*\nabla$, which corresponds to Eq. (4) in the case $G = \mathrm{SO}_n$ and $E = \Lambda^p T$.

Corollary 3.3. $\nabla^*\nabla = \sum_i T_i^* \circ T_i$

Let G be the holonomy group of an irreducible, non-symmetric Riemannian manifold. It is, then, well known that any isotypic component of $T \otimes E$ is irreducible, i.e. in the decomposition $T \otimes E = \sum E_i$ any summand E_i occurs only once (cf. [7]). As a consequence, the projection maps $\{\mathrm{pr}_i\}$ form a basis of $\mathrm{End}_G(T \otimes E)$ and any invariant endomorphism F of $T \otimes E$ can be written as $F = \sum f_i \mathrm{pr}_i$, with $F|_{E_i} = f_i \mathrm{id}$.

It turns out that for certain invariant endomorphisms F the operator $F \circ \nabla^2$ is in fact a zero order term. Hence, in these cases F gives rise to the Weitzenböck formula $F \circ \nabla^2 = -\sum f_i T_i^* T_i$. The following lemma provides us with a simple criterion for deciding which invariant endomorphisms F lead to Weitzenböck formulas.

Lemma 3.4. *Let F be an equivariant endomorphism of $T \otimes E$ considered as element of* $\mathrm{Hom}_G(T \otimes T, \mathrm{End}\, E)$. *Then, $F \circ \nabla^2$ defines a zero order operator if and only if $F_{a\otimes b} = -F_{b\otimes a}$ for any vectors $a, b \in T$.*

Proof. Let R be the curvature of EM and let $\{e_i\}$ be a parallel local frame, then:

$$F \circ \nabla^2 = \sum F(e_i \otimes e_j)\nabla_{e_i}\nabla_{e_j} = \frac{1}{2}\sum F(e_i \otimes e_j)(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})$$
$$= \frac{1}{2}\sum F(e_i \otimes e_j)R_{e_i,e_j}. \qquad \square$$

We show in Ref. [15] that $\mathrm{End}_G(T \otimes E)$ is in many cases, including the exceptional holonomies G_2 and Spin_7, the quotient of a polynomial algebra generated by one special endomorphism, the *conformal weight operator B*.

Definition 3.5. The conformal weight operator $B \in \mathrm{End}_G(T \otimes E) \cong \mathrm{Hom}_G(T \otimes T, \mathrm{End}\, E)$ is for any $a, b \in T$, $s \in E$ defined as:

$$B_{a\otimes b}s := \mathrm{pr}_{\mathfrak{g}}(a \wedge b)s,$$

where $\mathfrak{g}$ is the Lie algebra of the holonomy group G and $\mathrm{pr}_{\mathfrak{g}}$ denotes the projection $\Lambda^2 T \to \mathfrak{g} \subset \mathfrak{so}_n \cong \Lambda^2 T$. Here, $\mathfrak{g}$ acts via the differential of the representation π on E.

However, for the present article, it is only important to note that B defines a Weitzenböck formula, since obviously $B_{a\otimes b} = -B_{b\otimes a}$, for any $a, b \in T$. We will later apply this formula for proving that any Killing form on a compact manifold with exceptional holonomy has to be parallel.

The curvature term defined by B turns out to be the endomorphism $q(R)$ already introduced in Section 2. In fact, Eq. (5) can be considered as the Weitzenböck formula corresponding to B in the special case of $G = \mathrm{SO}_n$ and $E = \Lambda^p T$.

Lemma 3.6. $B \circ \nabla^2 = q(R)$

Proof. Let $\{X_i\}$ be an ortho-normal basis for the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$ and let $\{e_i\}$ be a local ortho-normal frame. Then:

$$B \circ \nabla^2 = \sum \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j)\nabla^2_{e_i,e_j} = \frac{1}{2}\sum \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j)(\nabla^2_{e_i,e_j} - \nabla^2_{e_j,e_i})$$
$$= \sum_{i<j} \mathrm{pr}_{\mathfrak{g}}(e_i \wedge e_j) R_{e_i,e_j} = \sum X_i \cdot R(X_i) = q(R). \qquad \square$$

In order to obtain the general Weitzenböck formula defined by B, we have to write $B = \sum b_i \mathrm{pr}_i$ and to determine the coefficients b_i. We, first, describe the conformal weight operator as an element of $\mathrm{End}_G(T \otimes E)$.

Lemma 3.7. *Let $\{X_i\}$ be an orthonormal basis for the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$. Then, $B = -\sum X_i \otimes X_i$, where X_i is acting on T resp. E via the holonomy representation resp. the representation E.*

Proof. Using the formula $\langle X, a \wedge b\rangle = \langle Xa, b\rangle$, for $a, b \in T$ and $X \in \Lambda^2 T \cong \mathfrak{so}_n$, we may write B as:

$$B(a \otimes s) = \sum e_i \otimes \mathrm{pr}_{\mathfrak{g}}(e_i \wedge a)s = \sum e_i \otimes \langle e_i \wedge a, X_j\rangle X_j s$$
$$= \sum \langle X_j e_i, a\rangle e_i \otimes X_j s = -\sum \langle e_i, X_j a\rangle e_i \otimes X_j s$$
$$= -\left(\sum X_j \otimes X_j\right) a \otimes s. \qquad \square$$

Let G be a compact semi-simple Lie group, with Lie algebra $\mathfrak{g}$ and let $\pi : G \to \mathrm{Aut}(V)$ be a representation of G on the complex vector space V. If $\{X_i\}$ is a basis of $\mathfrak{g}$, orthonormal with respect to an invariant scalar product g, the *Casimir operator* $\mathrm{Cas}^g_\pi \in \mathrm{End}(V)$ is defined as:

$$\mathrm{Cas}^g_\pi := \sum \pi_*(X_i) \circ \pi_*(X_i) = \sum X_i^2,$$

where $\pi_* : \mathfrak{g} \to \mathrm{End}(V)$ denotes the differential of the representation π. It is well-known that $\mathrm{Cas}^g_\pi = c^g_\pi \mathrm{id}_V$, if the representation π is irreducible. Moreover, the Casimir eigenvalues c^g_π can be expressed in terms of the highest weight of π.

It follows from the lemma above that the conformal weight operator B can be written as a linear combination of Casimir operators, which leads to the next corollary.

Corollary 3.8. *Let $T \otimes E = \oplus E_i$ be the decomposition of the tensor product into irreducible components. Then, the conformal weight operator B is given as:*

$$B = \sum b_i \mathrm{pr}_i \quad \text{with}\, b_i = \tfrac{1}{2}\left(c_T^{\Lambda^2} + c_E^{\Lambda^2} - c_{E_i}^{\Lambda^2}\right), \tag{7}$$

for Casimir eigenvalues $c_\pi^{\Lambda^2}$ computed with respect to the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$. The corresponding universal Weitzenböck formula on sections of EM is:

$$q(R) = -\sum b_i T_i^* T_i. \tag{8}$$

Proof. Expanding the Casimir operator $\mathrm{Cas}_{T\otimes E}^{\Lambda^2} = \sum X_i^2$ acting on $T \otimes E$, we obtain:

$$\mathrm{Cas}_{T\otimes E}^{\Lambda^2} = \sum \left(X_i^2 \otimes \mathrm{id}_E + 2X_i \otimes X_i + \mathrm{id}_T \otimes X_i^2\right).$$

Hence, Lemma 3.7 implies that the conformal weight operator can be written as:

$$B = -\tfrac{1}{2}\left(\mathrm{Cas}_{T\otimes E}^{\Lambda^2} - \mathrm{Cas}_T^{\Lambda^2} \otimes \mathrm{id}_E - \mathrm{id}_T \otimes \mathrm{Cas}_E^{\Lambda^2}\right),$$

which yields the formula above after restriction to the irreducible components E_i. □

Remark 3.9. In the case of Riemannian holonomy $G = \mathrm{SO}_n$ the Weitzenböck formula (8) was considered for the first time in Ref. [8]. In this article, one can also find the conformal weight operator and its expression in terms of Casimir operators. The operator B appears also in Ref. [6]. Similar results can be found in Ref. [9].

It remains to compute the Casimir eigenvalues. For doing so, we first, recall how to compute them for an irreducible representation of highest weight λ and with respect to the scalar product $(\cdot,\cdot)$ defined by the Killing form B. Let ρ be the half sum of the positive roots of $\mathfrak{g}$, then:

$$c_\pi^B = \|\rho\|^2 - \|\lambda + \rho\|^2 = -(\lambda, \lambda + 2\rho). \tag{9}$$

For the application of Corollary 3.8, we need the Casimir eigenvalues $c_\pi^{\Lambda^2}$ defined with respect to the induced scalar product on $\mathfrak{g} \subset \Lambda^2 T$. The relation between these Casimir eigenvalues is contained in the following normalization lemma, which we will apply in the case $V = T$.

Lemma 3.10. *Let $\mathfrak{g}$ be the Lie algebra of a compact simple Lie group and let V be an irreducible real $\mathfrak{g}$-representation with invariant scalar product $\langle\cdot,\cdot\rangle$. If π is any other irreducible $\mathfrak{g}$-representation with invariant scalar product g, then:*

$$c_\pi^{\Lambda^2} = -2\frac{\dim \mathfrak{g}}{\dim V}\frac{1}{c_V^g}c_\pi^g,$$

where $c_\pi^{\Lambda^2}$ denotes the Casimir eigenvalue with respect to the scalar product induced by $\langle\cdot,\cdot\rangle$ on $\mathfrak{g} \subset \mathfrak{so}(V) \cong \Lambda^2 V$. In particular, the Casimir eigenvalue of the representation V is given as $c_V^{\Lambda^2} = -2\frac{\dim \mathfrak{g}}{\dim V}$.

Proof. Since we assume V to be equipped with a invariant scalar product $\langle\cdot,\cdot\rangle$ we have $\mathfrak{g} \subset \mathfrak{so}(V) \cong \Lambda^2 V$. Restricting the induced scalar product onto $\mathfrak{g} \subset \Lambda^2 V$ defines the natural scalar product $\langle\cdot,\cdot\rangle_{\Lambda^2}$ on $\mathfrak{g}$. Note that $\langle\alpha,\beta\rangle_{\Lambda^2} = -\frac{1}{2}\mathrm{tr}_V(\alpha\circ\beta) = \frac{1}{2}\langle\alpha,\beta\rangle_{\mathrm{End}\,V}$. Let $\{X_a\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\cdot,\cdot\rangle_{\Lambda^2}$ and let $\{e_i\}$ be an orthonormal basis of V. Then, $\mathrm{Cas}_V^{\Lambda^2}(v) = c_V^{\Lambda^2} v = \sum_a X_a^2(v)$, for any $v \in V$, and we obtain:

$$\dim V c_V^{\Lambda^2} = \sum_{a,j} \langle X_a^2(e_j), e_j\rangle = -\sum_{a,j}\langle X_a(e_j), X_a(e_j)\rangle = -2\sum |X_a|^2_{\Lambda^2} = -2\dim \mathfrak{g},$$

which proves the lemma in the case $\pi = V$. Since $\mathfrak{g}$ is a simple Lie algebra it follows that two Casimir operators defined with respect to different scalar products differ only by a factor independent from the irreducible representation π. Hence, $\frac{c_\pi^{\Lambda^2}}{c_\pi^{g}} = \frac{c_V^{\Lambda^2}}{c_V^{g}}$ and the statement of the lemma follows from the special case $\pi = V$. □

In the remaining part of this section, we will consider the holonomy groups G_2 and Spin_7 and make the Weitzenböck formula (8) explicit for certain representations appearing in the decomposition of the form spaces.

3.1. The group G_2

The group $G_2 \subset \mathrm{SO}(7)$ is a compact simple Lie group of dimension 14 and of rank 2. As fundamental weights one usually considers ω_1 corresponding to the seven-dimensional holonomy representation T and ω_2 corresponding to the 14-dimensional adjoint representation $\mathfrak{g}_2$. The half-sum of positive roots is the sum of the fundamental weights, i.e. $\rho = \omega_1 + \omega_2$. Any other irreducible G_2-representation can be parameterized as $\Gamma_{a,b} = a\omega_1 + b\omega_2$, e.g. the trivial representation is $\Gamma_{0,0} = \mathbb{C}$. Further examples are:

$$\Gamma_{0,1} = \Lambda^2_{14} = \mathfrak{g}_2, \qquad \Gamma_{2,0} = \Lambda^3_{27}, \qquad \Gamma_{1,1} = V_{64}, \qquad \Gamma_{3,0} = V^-_{77},$$

where the subscripts denote the dimension of the representation, which is unique up to dimension 77. In dimension 77 on, it has two irreducible G_2-representations, denoted by V^+_{77} and V^-_{77}. Below, we need the following decomposition of the spaces of 2- and 3-forms:

$$\Lambda^2 T \cong \Lambda^5 T \cong T \oplus \Lambda^2_{14}, \qquad \Lambda^3 T \cong \Lambda^4 T \cong \mathbb{C} \oplus T \oplus \Lambda^3_{27}. \tag{10}$$

Since we want to apply the Weitzenböck formula for the bundles $\Lambda^2_{14}T$ and $\Lambda^3_{27}T$ we still need the following tensor product decompositions:

$$T \otimes \Lambda^2_{14} \cong T \oplus \Lambda^3_{27} \oplus V_{64}, \qquad T \otimes \Lambda^3_{27} \cong T \oplus \Lambda^4_{27} \oplus \Lambda^2_{14} \oplus V_{64} \oplus V^-_{77}. \tag{11}$$

There is a suitable invariant bilinear form g on $\mathfrak{g}_2$, which induces the scalar products:

$$g(\omega_1,\omega_1) = 1, \qquad g(\omega_2,\omega_2) = 3, \qquad g(\omega_1,\omega_2) = \tfrac{3}{2}.$$

The invariant bilinear form g is some multiple of the Killing form B and it follows for the Casimir eigenvalues that $c_\pi^g = \lambda c_\pi^B$, with some universal constant λ. This constant cancels in the formula for $c_\pi^{\Lambda^2}$ given in Lemma 3.10. Thus, we may use Eq. (9) and Lemma 3.10 for obtaining the following Casimir eigenvalues:

$$c^{\Lambda^2}_{\Gamma_{a,b}} = -\tfrac{2}{3}(a^2 + 3b^2 + 3ab + 5a + 9b).$$

In particular, we have, $c^{\Lambda^2}_{\Lambda^3_{27}} = -\frac{28}{3}$, $c^{\Lambda^2}_{\Lambda^2_{14}} = -8$, $c^{\Lambda^2}_{T} = -4$, $c^{\Lambda^2}_{V_{64}} = -14$, $c^{\Lambda^2}_{V^-_{77}} = -16$. The constant λ can be determined by noting that the Casimir eigenvalue of the adjoint representation computed with respect to the Killing form B is always -1.

Finally, we use (7) to obtain the Weitzenböck formula on the bundles Λ^2_{14} and Λ^3_{27}. Recall that the operator T_i is the projection of the covariant derivative onto the ith summand in the tensor product decomposition of $T \otimes E$, i.e. we will number the operators T_i according to the numbering of the summands in this decomposition, which has to be fixed in order to make the notation unique.

Here, we will consider the tensor product decomposition given in (11), e.g. in the case of the representation Λ^2_{14} the operator T_3 denotes the projection of the covariant derivative onto the summand V_{64}, whereas for the representation Λ^3_{27} the operator T_3 denotes the projection of the covariant derivative onto the summand Λ^2_{14}.

Proposition 3.11. *The operators T_i defined corresponding to the decompositions in* (11) *satisfy the following Weitzenböck formulas*:

$$\text{on} \quad \Lambda^2_{14} : q(R) = 4T_1^*T_1 + \tfrac{4}{3}T_2^*T_2 - T_3^*T_3,$$

$$\text{on} \quad \Lambda^3_{27} : q(R) = \tfrac{14}{3}T_1^*T_1 + 2T_2^*T_2 + \tfrac{8}{3}T_3^*T_3 - \tfrac{1}{3}T_4^*T_4 - \tfrac{4}{3}T_5^*T_5.$$

3.2. The group Spin$_7$

Let $\pm e_1, \pm e_2, \pm e_3$ and 0 be the weights of the seven-dimensional standard representation of Spin_7. Then, the fundamental weights may be expressed as:

$$\omega_1 = e_1, \qquad \omega_2 = e_1 + e_2, \qquad \omega_3 = \tfrac{1}{2}(e_1 + e_2 + e_3),$$

corresponding to the representations $\Lambda^1\mathbb{R}^7$, $\Lambda^2\mathbb{R}^7$ and the spin representation. All other irreducible Spin_7-representations are parameterized as $\Gamma_{a,b,c} = a\omega_1 + b\omega_2 + c\omega_3$. The half-sum of positive roots is given as $\rho = \omega_1 + \omega_2 + \omega_3 = \frac{5}{2}e_1 + \frac{3}{2}e_2 + \frac{1}{2}e_3$.

The holonomy group Spin_7 is considered as subgroup of SO_8 such that the holonomy representation T is given by the eight-dimensional spin representation, i.e. $T = \Gamma_{0,0,1}$. This leads to the following decompositions of the form spaces $\Lambda^k T$,

$$\Lambda^2 T \cong \Lambda^2_7 \oplus \Lambda^2_{21}, \qquad \Lambda^3 T \cong \Lambda^3_8 \oplus \Lambda^3_{48}, \qquad \Lambda^4 T \cong \Lambda^4_1 \oplus \Lambda^4_7 \oplus \Lambda^4_{27} \oplus \Lambda^4_{35}. \tag{12}$$

Again, the subscripts denote the dimensions of the representations and of course we have $\Lambda^2_7 \cong \Lambda^4_7$ and $\Lambda^3_8 = T$. For the investigation of forms on Spin_7-manifolds, we need Weitzenböck formulas on the bundles corresponding to Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. The decompositions of the tensor products $T \otimes E$ are given as:

$$\begin{aligned} &T \otimes \Lambda^2_{21} \cong T \oplus \Lambda^3_{48} \oplus V^a_{112}, \qquad T \otimes \Lambda^4_{27} \cong \Lambda^3_{48} \oplus V_{168}, \\ &T \otimes \Lambda^3_{48} \cong \Lambda^4_{35} \oplus \Lambda^2_{21} \oplus \Lambda^2_7 \oplus \Lambda^4_{27} \oplus V_{105} \oplus V_{189}, \\ &T \otimes \Lambda^4_{35} \cong T \oplus \Lambda^3_{48} \oplus V^a_{112} \oplus V^b_{112}. \end{aligned} \tag{13}$$

There are two 112-dimensional irreducible Spin_7-representation, which we denote with V^a_{112} and V^b_{112}. In terms of fundamental weights, the representations appearing in the above decompositions are given as follows:

$$\Lambda^2_7 = \Gamma_{1,0,0}, \quad \Lambda^2_{21} = \Gamma_{0,1,0}, \quad \Lambda^3_{48} = \Gamma_{1,0,1}, \quad \Lambda^4_{27} = \Gamma_{2,0,0}, \quad \Lambda^4_{35} = \Gamma_{0,0,2},$$

$$V^a_{112} = \Gamma_{0,1,1}, \quad V^b_{112} = \Gamma_{0,0,3}, \quad V_{168} = \Gamma_{2,0,1}, \quad V_{105} = \Gamma_{1,1,0}, \quad V_{189} = \Gamma_{1,0,2}.$$

Next, we have to calculate all the necessary Casimir eigenvalues. We choose on $\mathfrak{spin}_7$ an invariant scalar product g_0 such that the weights e_1, e_2, e_3 form an orthonormal base of the Lie algebra of the maximal torus, which is identified with $\mathbb{R}^3$. As in the G_2-case, we compute the Casimir eigenvalues using formula (9) and Lemma 3.10. However, in this case we have to take $V = \Gamma_{1,0,0} = \Lambda^2_7$, leading to $c_V^{\Lambda^2} = -6$ and:

$$c^{\Lambda^2}_{\Lambda^2_{21}} = -10, \quad c^{\Lambda^2}_T = -\tfrac{21}{4}, \quad c^{\Lambda^2}_{\Lambda^4_{27}} = -14, \quad c^{\Lambda^2}_{\Lambda^4_{35}} = -12, \quad c^{\Lambda^2}_{\Lambda^3_{48}} = -\tfrac{49}{4},$$

$$c^{\Lambda^2}_{V^a_{112}} = -\tfrac{69}{4}, \quad c^{\Lambda^2}_{V_{105}} = -18, \quad c^{\Lambda^2}_{V_{189}} = -20, \quad c^{\Lambda^2}_{V_{168}} = -\tfrac{85}{4}, \quad c^{\Lambda^2}_{V^b_{112}} = -\tfrac{81}{4}.$$

Finally, we use (7) to compute the coefficients of the Weitzenböck formula on the bundles Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. As in the G_2-case, we number the operators T_i corresponding to the decomposition (13), e.g. for the representation Λ^2_{21} the operator T_1 denotes the projection of the covariant derivative onto the summand T and for the representation Λ^3_{48} it denotes the projection of the covariant derivative onto the summand Λ^4_{35}.

Proposition 3.12. *Let $\{T_i\}$ be the operators defined corresponding to the decompositions in (13). Then, the following Weitzenböck formulas hold*:

$$\text{on} \quad \Lambda^2_{21} : q(R) = 5T_1^*T_1 + \tfrac{3}{2}T_2^*T_2 - T_3^*T_3,$$

$$\text{on} \quad \Lambda^3_{48} : q(R) = \tfrac{11}{4}T_1^*T_1 + \tfrac{15}{4}T_2^*T_2 + \tfrac{23}{4}T_3^*T_3 + \tfrac{7}{4}T_4^*T_4 - \tfrac{1}{4}T_5^*T_5 - \tfrac{5}{4}T_6^*T_6,$$

$$\text{on} \quad \Lambda^4_{27} : q(R) = \tfrac{7}{2}T_1^*T_1 - T_2^*T_2, \qquad \text{on} \quad \Lambda^4_{35} : q(R) = 6T_1^*T_1 + \tfrac{5}{2}T_2^*T_2 - \tfrac{3}{2}T_4^*T_4.$$

4. Proof of the theorems

In this section, we will prove Theorems 1.1 and 1.2 using the Weitzenböck formulas of Proposition 3.11 and 3.12. We will first show that on manifolds with holonomy G_2 and Spin_7 any Killing form can be decomposed into a sum of Killing forms belonging to the parallel sub-bundles of the form bundle. Hence, we may assume that the Killing form is a section of one of the irreducible components. The Weitenzböck formulas will then imply that all twistor operators vanish on the given Killing form, i.e. all components of the covariant derivative are zero and the Killing form has to be parallel. The statement for $*$-Killing forms is proved in a similar way.

4.1. The holonomy decomposition

Let (M^n, g) be a manifold with holonomy G, which is assumed to be a proper subgroup of SO_n. In this situation, the bundle of p-forms decomposes into a sum of parallel sub-bundles, $\Lambda^p TM = \oplus V_i$ and correspondingly, any p-form u has a holonomy decomposition $u = \sum u_i$. If u is a twistor form, or even a Killing form, it remains in general not true for the holonomy components u_i. Nevertheless, we have such a property in the case of the exceptional holonomies.

Lemma 4.1. *Let (M, g) be a compact manifold with holonomy G_2 or* Spin_7 *and let u be any form with holonomy decomposition $u = \sum u_i$. Then, u is a Killing form or a $*$-Killing form if and only if the same is true for all components u_i.*

Proof. We will use the characterization of Killing forms given in Proposition 2.2. Since the decomposition $\Lambda^p TM = \oplus V_i$ is parallel, it is preserved by $\nabla^*\nabla$ and $q(R)$. Thus, for a Killing p-form u all its holonomy components satisfy the equation $\nabla^*\nabla u_i = \frac{1}{p} q(R) u_i$ and it remains to verify whether the components are coclosed.

We start with the G_2-case and consider a Killing 2-form u with holonomy decomposition $u = u_7 + u_{14}$ according to (10). Since $q(R)$ acts as Ricci curvature on Λ^2_7, which vanishes for G_2-manifolds we have that $\nabla^*\nabla u_7 = 0$. Hence, u_7 is parallel and in particular coclosed. The same is then true for $u_{14} = u - u_7$. In the case of a Killing 3-form, we have the decomposition $u = u_1 + u_7 + u_{27}$ and, as for 2-forms, it follows that u_1 and u_7 have to be parallel, implying that u_{27} is coclosed. The argument is the same for Killing forms in degrees 4 and 5, since the same representations are involved. Finally, the proof for $*$-Killing forms follows from the duality under the Hodge star operator.

We, now, turn to the case of holonomy Spin_7. Let u be a 2-form with holonomy decomposition $u = u_7 + u_{21}$ or a 3-form with holonomy decomposition $u = u_8 + u_{48}$ according to (12). We showed in Proposition 2.4 that $q(R)$ acts trivially on u_7 and u_8. Thus, for a Killing form u these components are parallel and it follows, as in the G_2-case, that also the components u_{21} and u_{48} have to be coclosed.

It remains to consider the case of 4-forms. Here, it follows from Proposition 2.2 that a twistor 4-form u is characterized by the equation $\nabla^*\nabla u = \frac{1}{4} q(R) u$. Hence, all holonomy components of a twistor 4-form are again twistor 4-forms. Below, we will show that any twistor form in Λ^4_{27} has to be parallel. But then a Killing 4-form has three parallel holonomy components, implying as above that the fourth component has to be coclosed as well. $\square$

4.2. *Twistor forms on G_2-manifolds*

In this section, we will show that any Killing or $*$-Killing form u on a compact manifold of holonomy G_2 has to be parallel. By Lemma 4.1, we may assume that u is a section of one of the parallel sub-bundles of the form bundle. Moreover, we know already from Proposition 2.3 that every Killing or $*$-Killing form in a sub-bundle where $q(R)$ acts trivially has to be parallel. Hence, it remains to consider Killing or $*$-Killing forms in the sub-bundles Λ^2_{14} and Λ^3_{27}. According to the decomposition (11), we have three operators T_i in the first case and five in the second. We will show that for a Killing or $*$-Killing form all these operators have to vanish, such that the form has to be parallel. Some of the operators T_i vanish because of the twistor form condition, some since the form is assumed to be closed or coclosed and the remaining operators vanish because of the Weitzenböck formula of Proposition 3.11.

(1) The case Λ^2_{14}. The operator T_3 (with numeration according to decomposition (11)) vanishes on twistor forms, since the third summand V_{64} belongs neither to Λ^1 nor to Λ^3. Moreover, the differential splits as $d = d_7 + d_{27}$, e.g. $d_7 = \sum (e_i \wedge \nabla_{e_i})_7$, with subscripts denoting the projection onto the corresponding summand. There is no part d_1, since the trivial representation does not occur in the decomposition of $T \otimes \Lambda^2_{14}$. The projection pr_1 defining T_1 can be written in two ways:

$$\mathrm{pr}_1 : T \otimes \Lambda^2_{14} \xrightarrow{\pi_1} T \xrightarrow{j_1} T \otimes \Lambda^2_{14}, \qquad \mathrm{pr}_1 : T \otimes \Lambda^2_{14} \xrightarrow{\pi_2} \Lambda^3_7 \xrightarrow{j_2} T \otimes \Lambda^2_{14},$$

with $\pi_1(X \otimes \alpha) = X \lrcorner \alpha$ and $\pi_2(X \otimes \alpha) = (X \wedge \alpha)_7$, and the suitable right inverses j_1, j_2. Hence, $\mathrm{d}u = 0$ or $\mathrm{d}^* u = 0$ both imply $T_1 u = 0$. Similarly, $\mathrm{d}u = 0$ implies $T_2 u = 0$.

Let u be a $*$-Killing form in Λ^2_{14}, then, $\mathrm{d}u = 0$ implies $T_1 u = 0$ and $T_2 u = 0$. Hence, all twistor operators vanish on u and the form has to be parallel. Let u be a Killing form in Λ^2_{14}, then, only the component $T_2 u$ could be different from 0. But, the Weitzenböck formula of Proposition 3.11 and the equation, $2\nabla^*\nabla u = q(R) u$ imply $0 = 2\nabla^*\nabla u - q(R) u$

$= (2-\frac{4}{3})T_2^*T_2u$. Hence, $T_2^*T_2u = 0$, and after integration also $T_2u = 0$, i.e. the form u has to be parallel.

(2) The case Λ^3_{27}. Here, the operators T_4 and T_5 vanish on twistor forms, since the summands V_{64} and V_{77}^- of the decomposition (11) do not appear in the form spaces. Moreover, the differential splits as $\mathrm{d} = \mathrm{d}_7 + \mathrm{d}_{27}$ and the codifferential as $\mathrm{d}^* = \mathrm{d}^*_7 + \mathrm{d}^*_{14}$, again there is no component d_1. With the same arguments as for Λ^2_{14} we see that $\mathrm{d}u = 0$ implies $T_1u = 0$ and $T_2u = 0$ and $\mathrm{d}^*u = 0$ implies $T_1u = 0$ and $T_3u = 0$. Indeed, the first two summands of the decomposition (11) of $T \otimes \Lambda^3_{27}$ are also summands of the 4-forms. Similarly, the first and the third summand are components of the 2-forms.

Let u be Killing form in Λ^3_{27}, then only T_2u could be different from zero. But, the Weitzenböck formula and the equation $3\nabla^*\nabla u = q(R)u$ implies $(3-2)T_2^*T_2u = 0$ and u again has to be parallel. Finally, in the case of a $*$-Killing form u in Λ^3_{27}, we have to use the Weitzenböck formula and the equation $4\nabla^*\nabla u = q(R)u$ to show the vanishing of T_3u.

Remark 4.2. Using Lemma 3.2 and an explicit expressions for the projections onto the irreducible components of the form bundle it is possible to determine the precise relation between the operators $T_i^*T_i$ and similar operators in terms of the components of d and d^*, e.g. on the bundle Λ^2_{14} one finds: $\mathrm{d}^*_7\mathrm{d}_7 = T_1^*T_1$, $\mathrm{d}\mathrm{d}^* = 4T_1^*T_1$ and $3\mathrm{d}^*_{27}\mathrm{d}_{27} = 7T_2^*T_2$.

4.3. Twistor forms on Spin$_7$-manifolds

In this section, we will show that any Killing or $*$-Killing form u on a compact manifold of holonomy Spin_7 has to be parallel. Again, we may assume that u is a section of one of the parallel sub-bundles of the form bundle and we know already that every Killing or $*$-Killing form in a sub-bundle where $q(R)$ acts trivially has to be parallel. Hence, it remains in this case to consider Killing or $*$-Killing forms in the sub-bundles Λ^2_{21}, Λ^3_{48}, Λ^4_{27} and Λ^4_{35}. The argument is now similar to the G_2-case. For any Killing or $*$-Killing form u in one of these bundles, we show that all the operators T_i vanish on u, such that the form has to be parallel. In the Spin_7-case, the operators T_i are numbered according to the decomposition (13).

(1) The case Λ^2_{21}. According to the decomposition (13), we have three operators T_i in this case and T_3 vanishes on twistor forms since the third summand V^a_{112} belongs neither to Λ^1 nor to Λ^3. The representation $T \cong \Lambda^1$ appears also as summand in the 3-forms. Hence, as in the G_2-case, we see that $\mathrm{d}u = 0$ implies $T_1u = T_2u = 0$ and $\mathrm{d}^*u = 0$ implies $T_1u = 0$. Thus, $*$-Killing forms are automatically parallel. Let u be a Killing 2-form, then, $2\nabla^*\nabla u = q(R)u$ and the Weitzenböck formula of Proposition 3.12 imply $(2-\frac{3}{2})T_2^*T_2u$ and $T_2u = 0$. Thus, also, on Killing forms all operators T_i vanish.

(2) The case Λ^3_{48}. Here, we have six operators T_i and T_5, T_6 vanish on twistor forms since the corresponding summands V_{105} and V_{189} belong neither to Λ^2 nor to Λ^4. The representation Λ^2_7, i.e. the third summand in the decomposition (13) of $T \otimes \Lambda^3_{48}$, appears as summand in the 2- and 4-forms. Hence, $\mathrm{d}u = 0$ implies $T_1u = T_3u = T_4u = 0$ and $\mathrm{d}^*u = 0$ implies $T_2u = T_3u = 0$. Let u be a $*$-Killing form, then, $5\nabla^*\nabla u = q(R)u$ and we find $(5-\frac{15}{4})T_2^*T_2u = 0$, thus, $T_2u = 0$. Let u be a Killing form, then, $3\nabla^*\nabla u = q(R)u$ and $(3-\frac{11}{4})T_1^*T_1 + (3-\frac{7}{4})T_4^*T_4u = 0$ implies $T_1u = T_4u = 0$.

(3) The case Λ^4_{27}. Here, we have the operators T_1 and T_2 and, as above, T_2 vanishes on twistor forms. Let u be a twistor 4-form, i.e. $T_2 u = 0$. Then, $4\nabla^*\nabla u = q(R)u$ implies $(4 - \frac{7}{2})T_1^* T_1 u = 0$ and, thus, $T_1 u = 0$. Hence, any twistor 4-form in Λ^4_{27} has to be parallel, which completes the proof Lemma 4.1. Indeed, we had already seen that any holonomy component of a twistor 4-form is again a twistor form. Now, we see that three of the four components have to be parallel. Hence, for a Killing form all components are again coclosed, i.e. again Killing forms.
(4) The case Λ^4_{35}. Here, we have four operators T_i and T_3, T_4 vanish on twistor forms. Since both remaining summands T and Λ^3_{48} are sub-bundles of $\Lambda^3 \cong \Lambda^5$ we see that T_1 and T_2 vanish for closed or coclosed twistor forms, thus, they have to be parallel.

Remark 4.3. For the representation Λ^4_{35}, we find in Ref. [15] an additional Weitzenböck formula (with vanishing curvature term). This equation, then, shows that even any twistor 4-form on a Spin_7-manifold has to be parallel.

4.4. *Twistor forms*

In this last section, we will prove the more general statements on twistor forms, contained in Theorems 1.1 and 1.2.

It follows from Proposition 2.4 that on a compact G_2 or Spin_7 manifold any twistor 1-form, i.e. any conformal vector field, has to be parallel. Applying the Hodge star operator, we obtain that any twistor form of degree 6 resp. 7 has to be parallel.

Moreover, it is easy to show that on Einstein manifolds, and in particular on Ricci-flat manifolds, the codifferential of any twistor 2-form is either zero or dual to a Killing vector field. Hence, on a compact G_2 or Spin_7 manifold any twistor 2-form has to be coclosed and then also parallel. Applying again the Hodge star operator shows that any twistor form in degree 5 resp. 6 has to be parallel.

Summarizing, we note that arguments presented so for do not exclude the possibility of non-parallel twistor forms in degrees 3 and 4 for the G_2-case and in degrees 3–5 in the Spin_7-case.

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